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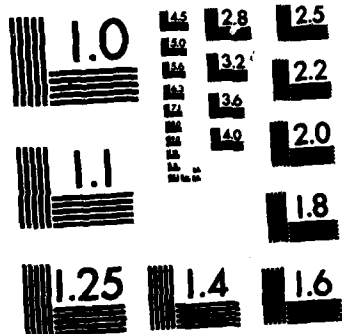
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Philip E. Gill, Walter Murray,  
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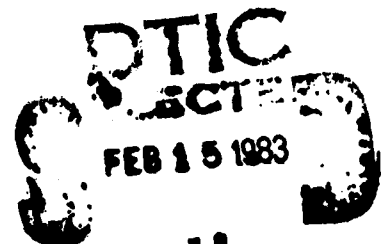
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# SPARSE MATRIX METHODS IN OPTIMIZATION†

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## ABSTRACT

Optimization algorithms typically require the solution of many systems of linear equations  $B_k x = b_k$ . When large numbers of variables or constraints are present, these linear systems could account for much of the total computation time.

Both direct and iterative equation solvers are needed in practice. Unfortunately, most of the off-the shelf solvers are designed for single systems, whereas optimization problems give rise to hundreds or thousands of systems. To avoid refactorization, or to speed the convergence of an iterative method, it is essential to note that  $B_k$  is related to  $B_{k-1}$ .

*The authors* We review various sparse matrices that arise in optimization, and discuss compromises that are currently being made in dealing with them. Since significant advances continue to be made with single-system solvers, *they* we give special attention to methods that allow such solvers to be used repeatedly on a sequence of modified systems (e.g., the product-form update; use of the Schur complement). The speed of factorizing a matrix then becomes relatively less important than the efficiency of subsequent solves with very many right-hand sides.

At the same time, *it is hoped* we hope that future improvements to linear-equation software will be oriented more specifically to the case of related matrices  $B_k$ .

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## 1. Introduction

**1.1. Background.** The major application of sparse matrix techniques in optimization up to the present has been in the implementation of the simplex method for linear programming (LP) (see, e.g., Dantsig, 1963). In fact, commercial codes for large LP problems seem to have predated codes for sparse linear equations (even though solving a sparse LP problem requires solving many sparse linear systems). In the commercial world today, more sparse matrix computation is probably expended on linear programs than on any other type of problem, and linear programs involving thousands of unknowns can be solved routinely. Because of the great success of the simplex algorithm and the wide availability of LP codes, many large-scale optimization problems tend to be formulated as purely linear programs. However, we shall see that this limitation is often unnecessary.

Before considering particular methods, we emphasize that methods for large-scale optimization have a special character attributable in large part to the critical importance of linear algebraic procedures. Since dense linear algebraic techniques tend to become unreasonably expensive as the problem dimension increases, it is usually necessary to compromise what seems to be an "ideal" strategy. (In fact, an approach that would not even be considered for small problems may turn out to be the best choice for some large problems.) Furthermore, the relative cost of the steps of many optimization methods changes when the problem becomes large. For example, the performance of unconstrained optimization algorithms is often measured by the number of evaluations of the objective function required for convergence. Although simplistic, this is a reasonable gauge of effectiveness for most problems of low dimension because the number of arithmetic operations per iteration tends to be small, and the amount of work required for storage manipulation is negligible. However, as the size of the problem grows, the "housekeeping" (cost of arithmetic and data structures) becomes comparable to, and may even dominate, the cost of function evaluations.

Most optimization methods are iterative; we shall consider algorithms in which the  $(k+1)$ -th iterate is defined as

$$x_{k+1} = x_k + \alpha_k p_k, \quad (1.1)$$

where  $\alpha_k$  is a non-negative scalar, and the  $n$ -vector  $p_k$  is called the search direction. One of the primary applications of sparse matrix techniques in optimization is in solving one or more systems of linear equations to obtain  $p_k$ .

It is usual for thousands of iterations to be required to solve a single large optimization problem, and hence it might appear that the computation time required would be enormous, even with the best available sparse matrix techniques. Fortunately, the linear systems that define  $p_{k+1}$  are usually closely related to those that define  $p_k$  (and the degree of closeness can be controlled to some extent by the choice of algorithm). In addition, the sequence  $\{x_k\}$  will often converge to the solution with only mild conditions on  $\{p_k\}$ . Consequently, there is a certain flexibility in the definition of  $p_k$ . The design of algorithms for large-scale optimization problems involves striking a balance between the effort expended at each iteration to compute  $p_k$  and the number of iterations required for convergence.

**1.2. Summary.** The three main subdivisions of optimization are discussed in turn (unconstrained, linearly constrained, and nonlinearly constrained). A common denominator is the need to solve many systems of linear equations, and the need to update various factorizations in order to deal with sequences of related equations. We indicate situations where off-the-shelf software can be applied. Symmetric positive-definite solvers are mainly useful for unconstrained problems, while unsymmetric solvers are essential for dealing with linear constraints. There is an inevitable

emphasis on the latter because most large optimisation problems currently being solved involve sparse linear constraints.

The principal updating problem is that of replacing one column of a square matrix. However, there exists only one generally available package for updating sparse factors *in situ*. We therefore focus on methods that allow an off-the-shelf solver to be used repeatedly on the same matrix with different right-hand sides. Such methods facilitate more general updates to sparse matrices. In one instance, a sparse indefinite solver is needed.

The final section on nonlinear constraints covers methods that solve a sequence of simpler subproblems, to which the preceding comments apply.

## 2. Unconstrained Optimization

**2.1. Methods for dense problems.** The unconstrained optimisation problem involves the minimisation of a scalar-valued objective function, i.e.

$$\underset{z \in \mathbb{R}^n}{\text{minimize}} \quad F(z).$$

We assume that  $F$  is smooth; let  $g(x)$  and  $H(x)$  denote the gradient vector and Hessian matrix of  $F$ .

Many techniques are available for solving unconstrained problems in which  $n$  is small (for recent surveys, see, e.g., Brodlie, 1977; Fletcher, 1980; Gill, Murray and Wright, 1981). The most popular methods compute the search direction as the solution of a system of linear equations of the form

$$H_k p_k = -g_k, \quad (2.1)$$

where  $g_k$  is the gradient of  $F$  at  $x_k$ , and  $H_k$  is a suitable symmetric matrix that is most often intended to represent (in some sense)  $H(x_k)$ . If  $H_k$  is positive definite, the solution of (2.1) is the step to the minimum of the local quadratic approximation to  $F$  at  $x_k$ :

$$\underset{p \in \mathbb{R}^n}{\text{minimize}} \quad g_k^T p + \frac{1}{2} p^T H_k p. \quad (2.2)$$

The major distinctions among algorithms involve the definition of  $H_k$ .

When  $H_k$  is the exact Hessian at  $x_k$  or a finite-difference approximation, the algorithm based on solving (2.1) for  $p_k$  is called a *Newton-type method*. Newton-type methods tend to be powerful and robust when properly implemented, and exhibit quadratic convergence under mild conditions. However, certain difficulties arise when  $H_k$  is indefinite, since the quadratic function (2.2) is unbounded below and the solution of (2.1) may be undefined. Numerous strategies have been suggested for this case, and often involve defining  $p_k$  as the solution of a linear system with a positive-definite matrix that is closely related to the Hessian. These techniques include the modified Cholesky factorisation of Gill and Murray (1974) and various trust-region strategies (see, e.g., Moré and Sorensen, 1982).

When an exact or finite-difference Hessian is unavailable or too expensive, a popular alternative is to use a *quasi-Newton method* (see Dennis and Moré, 1977, for a survey). In a quasi-Newton method, the matrix  $H_k$  is an approximation to the Hessian that is updated by a low-rank change at each iteration, based on information about the change in the gradient. The hope is that the approximation will improve as the iterations proceed. Quasi-Newton methods typically display a superlinear rate of convergence in practice, and are often more efficient (in terms of computation time) than Newton-type methods.

When  $n$  becomes very large, two related difficulties can occur with methods that solve (2.1) directly: excessive computation time and insufficient storage for the  $n \times n$  matrix  $H_k$ . Fortunately, the Hessian matrices of many large unconstrained problems are quite sparse, and density tends to decrease as  $n$  increases. Large problems can thus be solved efficiently using techniques that exploit sparsity in  $H_k$  to save work and/or storage, or that do not require storage of  $H_k$ .

**2.2. Newton-type methods.** When the Hessian is sparse and can be computed analytically, a Newton-type method can be implemented by applying standard sparse procedures to solve  $H_k p_k = -g_k$ . In particular, when  $H_k$  is positive definite, any efficient technique for computing a sparse Cholesky factorisation may be applied in this context (for a survey of available software, see Duff, 1982). Although many linear systems may need to be solved before the method converges, all of them have the same sparsity pattern, and hence the structure needs to be analyzed only once.

Indefiniteness in a sparse Hessian may be treated using the procedures mentioned for the dense case. The modified Cholesky factorization (Gill and Murray, 1974) has been adapted in a straightforward fashion to treat sparsity (see Thapa, 1980). One advantage of the modified Cholesky approach is that indefiniteness can be detected and corrected while constructing the factorization of the positive-definite matrix to be used in computing  $p_k$ ; hence, only one sparse factorization needs to be computed at each iteration. With trust-region methods,  $p_k$  may be obtained using off-the-shelf software for a sparse Cholesky factorization; however, these methods typically require more than one factorization per iteration.

When the gradient is available, but the exact Hessian is not, a finite-difference approximation to the Hessian may be used as  $H_k$ . In the general case, this requires  $n$  gradient evaluations. However, if the sparsity pattern of the Hessian is known *a priori*, it is possible to choose special vectors that allow a finite-difference approximation to  $H(x)$  to be computed with many fewer than  $n$  evaluations of the gradient.

For example, suppose that  $H(x)$  is tridiagonal:

$$H(x) = \begin{pmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ & & & \dots & \\ & & & & \times & \times & \times \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \end{pmatrix}.$$

Consider the vectors

$$y_i = \frac{1}{h\|x_i\|_2} (g(x_k + hx_i) - g(x_k)), \quad i = 1, 2,$$

where  $x_1 = (1, 0, 1, 0, \dots)^T$ ,  $x_2 = (0, 1, 0, 1, \dots)^T$ , and  $h$  is an appropriate finite-difference interval. Let  $y_{1,i}$  denote the  $i$ -th component of  $y_1$ , and similarly for  $y_2$ . The vectors  $y_1$  and  $y_2$  are approximations to the sums of odd and even columns of  $H_k$ , respectively. Therefore,

$$y_{1,1} \approx \frac{\partial^2 F}{\partial^2 x_1}; \quad y_{2,1} \approx \frac{\partial^2 F}{\partial x_1 \partial x_2}; \quad y_{1,2} \approx \frac{\partial^2 F}{\partial x_1 \partial x_2} + \frac{\partial^2 F}{\partial x_2 \partial x_3}; \quad \text{and so on.}$$

Thus, for example,

$$y_{1,2} - y_{2,1} \approx \frac{\partial^2 F}{\partial x_2 \partial x_3}.$$

In this fashion, all the elements of  $H_k$  can be approximated with only two evaluations of the gradient, regardless of the value of  $n$ .

The idea of analysing the sparsity pattern of the Hessian in order to determine suitable finite-difference vectors has been the subject of much recent interest. An algorithm for finding suitable finite-difference vectors for a sparse (unsymmetric) matrix is given by Curtis, Powell and Reid (1974), and is based on grouping together columns in which there are no overlapping elements. The problem of finding a minimal set of vectors can be viewed as a graph coloring problem in the directed graph that represents the sparsity pattern. A proof that finding the minimal set is NP-hard is given in Coleman and Moré (1981), along with practical algorithms (see also Coleman and Moré, 1982a).

A similar relationship with graph coloring can be developed for the case of a symmetric matrix; imposing the requirement of symmetry on the sparse matrix transforms the associated graph into a undirected graph. Although the problem of finding a minimal set is NP-complete (see McCormick, 1981), effective algorithms have been developed based on graph-theoretic heuristics. The algorithms are based on principles similar to those for the unsymmetric case, but are considerably complicated by exploiting symmetry.

A finite-difference Newton-type method for sparse problems thus begins with a procedure that analyses the sparsity pattern in order to determine suitable finite-difference vectors. Algorithms for finding these vectors have been given by Powell and Toint (1979) and Coleman and Moré (1982b). Once a sparse finite-difference Hessian approximation has been computed, a sparse factorization can be computed as with the exact Hessian.

**2.3. Sparse quasi-Newton methods.** Because of the great success of quasi-Newton methods on dense problems, it is natural to consider how such methods might be extended to take advantage of sparsity in the Hessian. This extension was suggested first for the case of sparse nonlinear equations by Schubert (1970), and was analyzed by Marwil (1978). Discussions of sparse quasi-Newton methods for optimization and nonlinear equations are given in Toint (1977), Dennis and Schnabel (1979), Toint (1979), Shanno (1980), Steihaug (1980), Thapa (1980), Powell (1981), Dennis and Marwil (1982) and Sorensen (1982). In the remainder of this section we give a brief description of sparse quasi-Newton methods applied to unconstrained optimization.

In quasi-Newton methods for dense problems, the Hessian approximation  $H_k$  is updated at each iteration by the relationship

$$H_{k+1} = H_k + U_k.$$

The update matrices  $U_k$  associated with many dense quasi-Newton methods are of rank two, and can be shown to be the minimum-norm symmetric change in  $H_k$ , subject to satisfying the quasi-Newton condition

$$H_{k+1}s_k = y_k, \quad (2.3)$$

where  $s_k = x_{k+1} - x_k$  and  $y_k = g_{k+1} - g_k$  (see, e.g., Dennis and Moré, 1977). By suitable choice of the steplength  $\alpha_k$  in (1.1), the property of hereditary positive-definiteness can also be maintained (i.e.,  $H_{k+1}$  is positive definite if  $H_k$  is). However, the update matrices  $U_k$  do not retain the sparsity pattern of the Hessian.

The initial approach to developing sparse quasi-Newton updates was to impose the additional constraint of retaining sparsity on the norm-minimisation problem (Powell, 1976; Toint, 1977). Let  $\mathcal{N}$  be defined as the set of indices  $\{(i, j) \mid H_{ij}(x) = 0\}$ , so that  $\mathcal{N}$  represents the specified sparsity pattern of the Hessian, and assume that  $H_k$  has the same sparsity pattern. A sparse

update matrix  $U_k$  is then the solution of

$$\begin{aligned} & \underset{U}{\text{minimize}} && \|U\| \\ & \text{subject to} && (H_k + U)s_k = y_k \\ & && U = U^T, \\ & && U_{ij} = 0 \text{ for } (i, j) \in \mathcal{N}. \end{aligned} \quad (2.4)$$

Let  $\sigma^{(j)}$  denote the vector  $s_k$  with the sparsity pattern of the  $j$ -th column of  $H_k$  imposed. When the norm in (2.4) is the Frobenius norm, the solution is given by

$$U_k = \sum_{j=1}^n \lambda_j (e_j \sigma^{(j)T} + \sigma^{(j)} e_j^T), \quad (2.5)$$

where  $e_j$  is the  $j$ -th unit vector and  $\lambda$  is the vector of Lagrange multipliers associated with the subproblem (2.4). The vector  $\lambda$  is the solution of the linear system

$$Q\lambda = y_k - H_k s_k, \quad (2.6)$$

where

$$Q = \sum_{j=1}^n (\sigma_j^{(j)} \sigma^{(j)} + \|\sigma^{(j)}\|_2^2 e_j e_j^T).$$

The matrix  $Q$  is symmetric and has the same sparsity pattern as  $H_k$ ;  $Q$  is positive definite if and only if  $\|\sigma^{(j)}\| > 0$  for all  $j$ . (The sparse analogue of any quasi-Newton formula may be obtained using a similar analysis; see Shanno, 1979, and Thapa, 1980).

Thus far, sparse quasi-Newton methods have not enjoyed the great success of their dense counterparts. First, there are certain complications that result from the requirement of sparsity. In particular, note that the update matrix  $U_k$  (2.5) is of rank  $n$ , rather than of rank two; this means that the new approximate Hessian cannot be obtained by a simple update of the previous approximation. Second, an additional sparse linear system (2.6) must be solved in order to compute the update. Finally, it is not possible in general to achieve the property of hereditary positive-definiteness in the matrices  $\{H_k\}$  if the quasi-Newton condition is satisfied (see Toint, 1979, and Sorensen, 1982); in fact, positive-definiteness may not be retained even if  $H_k$  is taken as the exact (positive definite) Hessian and the initial  $x_k$  is very close to the solution (see Thapa, 1980).

In addition to these theoretical difficulties, computational results have tended to indicate that currently available sparse quasi-Newton methods are less effective than alternative methods (in terms of the number of function evaluations required for convergence). However, hope remains that their efficiency may be improved — for example, by relaxing the quasi-Newton condition (2.3), or by finding only an approximate solution of (2.6) (Steihaug, 1982). For a discussion of some possible new approaches, see Sorensen (1982).

**2.4. Conjugate-gradient methods.** The term *conjugate-gradient* refers to a class of optimisation algorithms that generate directions of search without storing a matrix. They are essential in circumstances when methods based on matrix factorisation are not viable because the relevant matrix is too large or too dense. We emphasize that there are two types of conjugate-gradient method — linear and nonlinear.

The linear conjugate-gradient method was originally derived as an iterative procedure for solving positive-definite symmetric systems of linear equations (Hestenes and Stiefel, 1952). It has been studied and analyzed by many authors (see, e.g., Reid, 1971). When applied to the positive-definite symmetric linear system

$$Hx = -c, \quad (2.7)$$

it computes a sequence of iterates using the relation (1.1). The vector  $p_k$  is defined by

$$p_k = -(Hx_k + c) + \beta_{k-1}p_{k-1}, \quad (2.8)$$

and the step length  $\alpha_k$  is given by an explicit formula. The matrix  $H$  need not be stored explicitly, since it appears only in matrix-vector products.

With exact arithmetic, the linear conjugate-gradient algorithm will compute the solution of (2.7) in at most  $m$  ( $m \leq n$ ) iterations, where  $m$  is the number of distinct eigenvalues of  $H$ . Therefore, the number of iterations required should be significantly reduced if the original system can be replaced by an equivalent system in which the matrix has clustered eigenvalues. The idea of *preconditioning* is to construct a transformation to have this effect on  $H$ . One of the earliest references to preconditioning for linear equations is Axelsson (1974). See Concus, Golub and O'Leary (1976) for details of various preconditioning methods derived from a slightly different viewpoint.

The *nonlinear* conjugate-gradient method is used to minimize a nonlinear function without storage of any matrices, and was first proposed by Fletcher and Reeves (1964). In the Fletcher-Reeves algorithm,  $p_k$  is defined as in the linear case by (2.8), where the term  $Hx_k + c$  is replaced by  $g_k$ , the gradient at  $x_k$ . For a nonlinear function,  $\alpha_k$  in (1.1) must be computed by an iterative step-length procedure. When the initial vector  $p_0$  is taken as the negative gradient and  $\alpha_k$  is the step to the minimum of  $F$  along  $p_k$ , it can be shown that each  $p_k$  is a direction of descent for  $F$ .

Many variations and generalizations of the nonlinear conjugate-gradient method have been proposed. The most notable features of these methods are:  $\beta_k$  is computed using different definitions;  $p_k$  is defined as a linear combination of several previous search directions;  $p_0$  is not always chosen as the negative gradient; and  $\alpha_k$  is computed with a relaxed linear search (i.e.,  $\alpha_k$  is not necessarily a close approximation to the step to the minimum of  $F$  along  $p_k$ ). Furthermore, the idea of preconditioning may be extended to nonlinear problems by allowing a preconditioning matrix that varies from iteration to iteration.

It is well known that rounding errors may cause even the linear conjugate-gradient method to converge very slowly. The nonlinear conjugate-gradient method displays a range of performance that has not yet been adequately explained. On problems in which the Hessian at the solution has clustered eigenvalues, a nonlinear conjugate-gradient method will sometimes converge more quickly than a quasi-Newton method, whereas on other problems the method will break down, i.e. generate search directions that lead to essentially no progress. For recent surveys of conjugate-gradient methods, see Gill and Murray (1979), Fletcher (1980) and Hestenes (1980).

**2.5. The truncated linear conjugate-gradient method.** Much recent interest has been focussed on an approach to unconstrained optimisation in which the equations (2.1) that define the search direction are "solved" (approximately) by performing a limited number of iterations of the linear conjugate-gradient method.

Consider the case in which the exact Hessian is used in (2.1). Dembo, Eisenstat and Steihaug (1982) note that the local convergence properties of Newton's method depend on  $p_k$  being an

accurate solution of (2.1) *only near the solution of the unconstrained problem*. They present a criterion that defines the level of accuracy in  $p_k$  necessary to achieve quadratic convergence as the solution is approached, and suggest systematically "truncating" the sequence of linear conjugate-gradient iterates when solving the linear system (2.1) (hence their name of "truncated Newton method"). (See also Dembo and Steihaug, 1980, and Steihaug, 1980.)

This idea has subsequently been applied in a variety of situations — for example, in computing a search direction from (2.1) when  $H_k$  is a sparse quasi-Newton approximation (Steihaug, 1982). We therefore prefer the more specific name of *truncated conjugate-gradient methods*. These methods are useful in computing search directions when it is impractical to store  $H_k$ , but it is feasible to compute a *relatively small* number of matrix-vector products involving  $H_k$ . For example, this would occur if  $H_k$  were the product of several sparse matrices whose product is dense (see Section 3.3.1). Truncated conjugate-gradient methods have also been used when the matrix-vector product  $H_k v$  is approximated (say, by a finite-difference along  $v$ ); in this case, the computation of  $p_k$  requires a number of gradient evaluations equal to the number of linear conjugate-gradient iterations (see, e.g., O'Leary, 1982). In order for these methods to be effective, it must be possible to compute a good solution of (2.1) in a small number of linear conjugate-gradient iterations, and hence the use of preconditioning is important.

With a truncated conjugate-gradient method, complications arise when the matrix  $H_k$  is not positive definite, since the linear conjugate-gradient method is likely to break down. Various strategies are possible to ensure that  $p_k$  is still a well defined descent direction even in the indefinite case. For example, the conjugate-gradient iterates may be computed using the Lanczos process (Paige and Saunders, 1975); a Cholesky factorization of the resulting tridiagonal matrix leads to an algorithm that is equivalent to the usual iteration in the positive-definite case. If the tridiagonal matrix is indefinite, a related positive-definite matrix can be obtained using a modified Cholesky factorization. Furthermore, preconditioning can be included, in which case the linear conjugate-gradient iterates begin with the negative gradient transformed by the preconditioning matrix. If the preconditioning matrix is a good approximation to the Hessian, the iterates should converge rapidly. Procedures of this type are described in O'Leary (1982), Nash (1982), and Gill et al. (1983).

Further flexibility remains as to how the result of a truncated conjugate-gradient procedure may be used within a method for unconstrained optimization. Rather than simply being used as a search direction, for example,  $p_k$  may be combined with previous search directions in a *nonlinear* conjugate-gradient method (see Nash, 1982, and Gill et al., 1983).

### 3. Linearly Constrained Optimization

**3.1. Introduction.** The linearly constrained problem will be formulated as

LCP	$\begin{aligned} &\text{minimize} && F(x) \\ &\quad \quad \quad x \in \mathbb{R}^n \\ &\text{subject to} && Ax = b \\ &\quad \quad \quad l \leq x \leq u \end{aligned}$
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where the  $m \times n$  matrix  $A$  is assumed to be large and sparse. For simplicity, we assume that the rows of  $A$  are linearly independent (if not, some of them may be removed without altering the solution).

The most popular methods for linearly constrained optimization are *active-set* methods, in which a subset of the constraints (the *working set*), is used to define the search direction. The

working set at  $x_k$  usually includes constraints that are satisfied exactly at  $x_k$ ; the search direction is then computed so that movement along  $p_k$  will continue to satisfy the constraints in the working set.

With problem LCP, the working set will include the general constraints  $Ax = b$  and some of the bounds. When a bound is in the working set, the corresponding variable is fixed during that iteration. Thus, the working set induces a partition of  $x$  into fixed and free variables.

We shall not be concerned with details of how the working set is altered, but merely emphasize that the fixed variables at a given iteration are effectively removed from the problem; the corresponding components of the search direction will be zero, and thus the columns of  $A$  corresponding to fixed variables may be ignored. Let  $A_k$  denote the submatrix of  $A$  corresponding to the free variables at iteration  $k$ ; each change in the working set corresponds to a change in the columns of  $A_k$ . Let  $n_v$  denote the number of free variables, and the vector  $p_k$  denote the search direction with respect to the free variables only.

By analogy with (2.2) in the unconstrained case, we may choose  $p_k$  as the step to the minimum of a quadratic approximation to  $F$ , subject to the requirement of remaining on the constraints in the working set. This gives  $p_k$  as the solution of the following quadratic program:

$$\begin{aligned} & \underset{p}{\text{minimize}} && g_k^T p + \frac{1}{2} p^T H_k p \\ & \text{subject to} && A_k p = 0, \end{aligned} \quad (3.1)$$

where  $g_k$  denotes the gradient and  $H_k$  the Hessian (or Hessian approximation) at  $x_k$  with respect to the free variables.

The solution  $p_k$  and Lagrange multiplier  $\lambda_k$  of the problem (3.1) satisfy the  $n_v + m$  equations

$$\begin{pmatrix} H_k & A_k^T \\ A_k & \end{pmatrix} \begin{pmatrix} p_k \\ -\lambda_k \end{pmatrix} = \begin{pmatrix} -g_k \\ 0 \end{pmatrix}, \quad (3.2)$$

which will be called the *augmented system*.

One convenient way to represent  $p_k$  involves a matrix whose columns form a basis for the null space of  $A_k$ . Such a matrix, which will be denoted by  $Z_k$ , has  $n_v - m$  linearly independent columns and satisfies  $A_k Z_k = 0$ . The solution of (3.1) may then be computed by solving the *null-space equations*

$$Z_k^T H_k Z_k p_z = -Z_k^T g_k \quad (3.3)$$

and setting

$$p_k = Z_k p_z. \quad (3.4)$$

Equations (3.3) and (3.4) define a *null-space* representation of  $p_k$  (so named because it explicitly involves  $Z_k$ ). The vector  $Z_k^T g_k$  and the matrix  $Z_k^T H_k Z_k$  are called the *projected gradient* and *projected Hessian*.

**3.2. Representation of the null space.** The issues that arise in representing  $Z_k$  when  $A_k$  is sparse illustrate the need to compromise strategies that are standard for dense problems. In the rest of this section, we shall drop the subscript  $k$  associated with the iteration.

In dense problems, it is customary to use an explicit  $LQ$  or some other orthogonal factorisation of  $A$  in order to define  $Z$ . If  $AQ = \begin{pmatrix} L & 0 \end{pmatrix}$ , where the orthonormal matrix  $Q$  is partitioned as  $\begin{pmatrix} Y & Z \end{pmatrix}$  and  $L$  is lower triangular, then  $AZ = 0$ . In this case,  $Z$  has the "ideal" property

that its columns are orthonormal, so that formation of the projected Hessian and gradient does not exacerbate the condition of (3.3) and (3.4). Unfortunately, for large problems computation of such a factorization is normally too expensive. (Some current research is concerned with efficient methods for obtaining sparse orthogonal factorizations; see George and Heath, 1981. However, the need to update the factors is an even more serious difficulty; see Heath, 1982, and George and Ng, 1982.)

If an orthogonal factorization is unacceptable, a good alternative is to reduce  $A$  to triangular form using Gaussian elimination (i.e., elementary transformations combined with row and column interchanges). This would give an  $LU$  factorization in the form

$$P_1 A P_2 \begin{pmatrix} U & W \\ & I \end{pmatrix} = (L \ 0), \quad (3.5)$$

where  $P_1$  and  $P_2$  are permutation matrices,  $U$  is unit upper triangular, and  $L$  is lower triangular. The matrices  $P_1$  and  $P_2$  would be chosen to make  $U$  well conditioned and  $\|W\|$  reasonably small. The required matrix

$$Z = P_2 \begin{pmatrix} W \\ I \end{pmatrix} \quad (3.6)$$

would no longer have orthonormal columns, but should be quite well conditioned, even if  $A$  is poorly conditioned.

Unfortunately, it is not known how to update the factorization (3.5) efficiently in the sparse case when columns of  $A$  are altered. However, (3.5) indicates the existence of a square, nonsingular submatrix drawn from the rows and columns of  $A$ . We shall assume for simplicity that this matrix comprises the left-most columns of  $A$ , i.e.

$$A = (B \ S), \quad (3.7)$$

where  $B$  is non-singular. (In practice, the columns of  $B$  may occur anywhere in  $A$ .) It follows from (3.7) and (3.5) (with  $P_1$  and  $P_2$  taken as identity matrices) that  $BW + S = 0$ , so that  $W = -B^{-1}S$ . Thus,  $Z$  has the form

$$Z = \begin{pmatrix} -B^{-1}S \\ I \end{pmatrix}. \quad (3.8)$$

As long as  $B$  in (3.7) is nonsingular, the matrix  $Z$  (3.8) will provide a basis for the null space of  $A$ . In the absence of the ideal factorization (3.5), the aim must be to choose a  $B$  that is as well conditioned as conveniently possible, since this will tend to limit the size of  $\|W\|$  and hence the condition of  $Z$ .

The partition of the columns of  $A$  given by (3.7) induces a partition of the free variables, which will be indicated by the subscripts "b" and "s". The  $m$  variables  $x_b$  are called the *basic* variables. The remaining  $s$  free variables ( $s = n_v - m$ ) are called the *superbasic* variables. For historical reasons, the fixed variables are sometimes called the *nonbasic* variables.

An advantage of the form (3.8) for sparse problems is that operations with  $Z$  and  $Z^T$  may be performed using a factorization of the matrix  $B$ ; the matrix  $Z$  itself need not be stored. For example, the vector  $Z^T g$  required in (3.3) may be written as

$$Z^T g = -S^T B^{-T} g_b + g_s. \quad (3.9)$$

(The vector on the right-hand side of (3.9) is called the *reduced gradient*; note that it is simply the projected gradient with a particular form of  $Z$ .) Thus,  $Z^T g$  may be obtained by solving  $B^T v = g$ , and then forming  $g - S^T v$ . Similarly, to form  $p = Z p_r$ , we have

$$p = \begin{pmatrix} -B^{-1}S \\ I \end{pmatrix} p_r = \begin{pmatrix} -B^{-1}S p_r \\ p_r \end{pmatrix},$$

which gives the system

$$B p_r = -S p_r.$$

With the *reduced-gradient form of  $Z$*  (3.8), the problems of representing a null space and computing the associated projections reduce to the familiar operations of factorizing and solving with an appropriate square  $B$ .

**3.3. Solving for the search direction.** At each iteration of an active-set method for LCP, the search direction  $p$  with respect to the free variables solves the subproblem (3.1). We have seen that there are mathematically equivalent representations of  $p$ ; the way in which  $p$  is computed for sparse problems depends on several considerations, which will be discussed below.

**3.3.1. Solving the null-space equations.** For sparse problems, it will generally not be possible to solve (3.3) by explicitly forming and then factorizing  $Z^T H Z$ . Even if  $H$  and  $B$  are sparse, the projected Hessian will generally be dense. Thus, if a factorization of the projected Hessian is to be stored, the number of superbasic variables at each iteration must be sufficiently small (i.e., the number of fixed variables must be sufficiently large). Fortunately, for many large-scale problems there is an *a priori* upper bound on the number of free variables. For example, if only  $q$  of the variables appear nonlinearly in the objective function, the dimension of the projected Hessian matrix at the solution cannot exceed  $q$ .

Furthermore, even if the dimension of  $Z^T H Z$  is small, forming the projected Hessian may involve a substantial amount of work; when  $Z$  is defined by (3.8), computation of  $Z^T H Z$  requires the solution of  $2s$  systems of size  $m \times m$ . For this reason, a Newton-type method in which the projected Hessian is recomputed at each iteration is not generally practical. By contrast, *quasi-Newton methods* can be adapted very effectively to sparse problems in which the dimension of the projected Hessian remains small, by updating a dense Cholesky factorization of a quasi-Newton approximation to the *projected Hessian*; this is the method used in the MINOS code of Murtagh and Saunders (1977, 1980).

When the projected Hessian cannot be formed or factorized, the null-space equations may be solved using an iterative method that does not require storage of the matrix, such as a truncated conjugate-gradient method (see Section 2.5). In order for this approach to be reasonable, the computation of matrix-vector products involving  $Z$  and  $H$  must be relatively cheap (e.g., when  $H$  is sparse); in addition, a good approximation to the solution of (3.3) must be obtained in a small number of iterations. Even when the Hessian is not available, a truncated conjugate-gradient method may be applied to (3.3) by using a finite-difference of the gradient to approximate the vector  $H Z v$ ; an evaluation of the gradient is thus necessary for every iteration of the truncated conjugate-gradient method. Note that this is one of the few methods in which  $H$  is not required to be sparse. Some experience with a truncated conjugate-gradient approach in this context is described in Gill, et al., 1983.

Each of the above methods for solving the null-space equations can be adapted to allow for changes in the working set (Section 3.5).

**3.3.2. Solving the range-space equations.** The null-space equations provide one means of solving for  $p$  in the augmented system (3.2), by eliminating  $\lambda_k$ . When  $H$  is positive definite, a complementary approach is to solve for  $\lambda$  first, via the *range-space equations*

$$\begin{aligned} AH^{-1}A^T\lambda &= AH^{-1}g, \\ Hp &= A^T\lambda - g. \end{aligned}$$

This method would be appropriate if  $H$  were sparse, and if  $A$  had relatively few rows. The application of a range-space approach to quadratic programming is discussed by Gill *et al.* (1982).

**3.3.3. Solving the augmented system.** An alternative method for obtaining  $p$  involves treating the augmented system directly. (Variations of this idea have been proposed by numerous authors; see, e.g., Bartels, Golub and Saunders, 1970). The most obvious way to solve (3.2) is to apply a method for symmetric indefinite systems, such as the Harwell code MA27 (Duff and Reid, 1982). In order for the solution of (3.2) to be meaningful, the matrix  $Z^THZ$  must be positive definite. Verifying positive-definiteness in this situation is a nontrivial task, since of course the matrix  $Z^THZ$  is not computed explicitly. However, the result may sometimes be known *a priori* — for example, when  $H$  itself is positive-definite.

Both  $H$  and  $A$  change dimension when the working set is altered. Updating procedures for this case are discussed in Section 3.6.2.

**3.4. Factorizing and solving a square system.** The linear systems involving  $B$  and  $B^T$  are typically solved today using a sparse  $LU$  factorization of  $B$ . Surveys of techniques for computing such a factorization are given in Duff (1982) and Duff and Reid (1983). The *analyze phase* of a factorization consists of an analysis of the sparsity pattern alone (independent of the values of the elements), and leads to a permutation of the matrix in order to reduce fill-in during the factorization. The *factor phase* consists of computation with the actual numerical elements of the matrix.

We shall mention a few features of certain factorization methods that have particular relevance to optimization (see Duff and Reid, 1983, for more details). Since active-set algorithms include a sequence of matrices that undergo column changes, the factorization methods were typically developed to be used in conjunction with an update procedure.

The  $P^4$  algorithm of Hellerman and Rarick (1971, 1972) performs the analyze phase separately from the factor phase, and produces the well known "bump and spike" structure, in which  $B$  is permuted to block lower-triangular form with relatively few "spikes" (columns containing nonzeros above the diagonal). This procedure is very effective if  $B$  is nearly triangular. Also, the factor phase is able to use external storage, since it processes  $B$  one column at a time. Column interchanges are used to stabilize the factorization. (Row interchanges would destroy the sparsity pattern.) If an interchange is needed at the  $i$ -th stage, it is necessary to solve a system of the form  $L_{i-1}^T y = e_i$ , and to compute the quantities  $y^T a_j$  for all remaining eligible spike columns  $a_j$ . This involves significant work and also degrades the sparsity of the factors. Thus, a rather loose pivot tolerance must be used to avoid many column interchanges (e.g.,  $|\mu| \leq 10^4$ , where  $\mu$  is the largest subdiagonal element in any column of  $L$  divided by the corresponding diagonal).

The Markowitz algorithm (Markowitz, 1957), on the other hand, performs the analyze and factor phases simultaneously, and hence must run in main memory. It computes dynamic "merit counts" in order to determine the row and column permutations to preserve sparsity and yet retain numerical stability. The Markowitz procedure can achieve a good sparse factorization even with a rather strict pivot tolerance (e.g.,  $|\mu| \leq 10$ ).

Table 1

## Summary of Problem Characteristics

	Stair 1	Stair 2	Stair 3	OPF 1	OPF 2
<i>B</i> rows	357	745	1170	1200	3400
<i>B</i> nonzeros	3500	3600	7100	9000	29000
$P^4$ blocks	1	5	13	1	—
$P^4$ spikes	66	101	157	715	—

Table 2

Number of Nonzeros in initial *LU* factorisation and after *k* updates

	Stair 1	Stair 2	Stair 3	OPF 1	OPF 2
$L_0U_0$ with $P^4$ (MINOS)	9400	16200	32000	30400	—
$L_0U_0$ with Markowitz (LA05)	5400	4700	13500	13800	75000
<i>k</i>	50	50	50	30	40
$L_kU_k$ with LA05	7800	6000	17100	15300	83000

In order to indicate how these factor routines perform on matrices that arise in optimisation, we give results on five test problems. In the first three problems, the matrix *B* has "staircase" structure (see, e.g., Fourer, 1982); constraints of this form often arise in the modeling of dynamic systems, in which a set of activities is replicated over several time periods. The fourth and fifth problems arise from the optimal power flow (OPF) problem (see, e.g., Stott, Alsac and Marinho, 1980). In this case, *B* is the Jacobian of the network equations of the power system, and has a symmetric sparsity pattern (which is not at all triangular!) Table 1 shows some of the relevant features of the problems described, including the results of factorisation with the  $P^4$  algorithm.

The number of nonzeros in the initial *LU* factorisation of *B* is shown in the first two rows of Table 2. The  $P^4$  algorithm is as implemented in the MINOS code of Murtagh and Saunders

(1977, 1980); the Markowitz procedure is the Harwell code LA05 (Reid, 1976, 1982). Note that the large number of spikes in the first OPF problem is bound to cause difficulties for the  $P^4$  algorithm.

### 3.5. Column updates

For problems of the form LCP, each change in the working set involves changing the status of a variable from fixed to free (or vice versa). When a previously fixed variable becomes freed, a column of  $A$  is added to  $A$ ; this poses no particular difficulty, since the new column can simply be appended to  $S$ . When a free variable is to become fixed, a column of  $A$  must be deleted, and complications arise if the column is in  $B$ . Since the number of columns in  $B$  must remain constant (in order for  $B$  to be nonsingular), it is necessary to replace a column of  $B$  with one of the columns of  $S$ .

Assume that we are given an initial  $B_0$ , which thereafter undergoes a sequence of column replacements, each corresponding to one of the free variables becoming fixed on a bound. Let  $l_k$  denote the index of the column to be replaced at the  $k$ -th step,  $a_k$  denote the  $l_k$ -th column of  $B$ ,  $v_k$  denote the new column, and  $e_{l_k}$  denote the  $l_k$ -th column of the identity matrix. After each replacement, we have

$$B_k = B_{k-1} + (v_k - a_k)e_{l_k}. \quad (3.10)$$

We shall consider several ways in which systems of equations involving  $B_k$  can be solved following a sequence of such changes.

**3.5.1. The product-form update.** The standard updating technique used in all early sparse LP codes was the product-form (PF) update (e.g., Dantsig and Orchard-Hays, 1954). It follows from the definition of  $B_k$  that

$$B_k = B_{k-1}T_k,$$

where

$$B_{k-1}y_k = v_k \quad \text{and} \quad T_k = I + (y_k - e_{l_k})e_{l_k}^T. \quad (3.11)$$

Note that  $T_k$  is a permuted triangular matrix (with only one nontrivial column); equivalently,  $T_k$  is a rank-one modification of the identity matrix. The matrix  $T_k$  can be represented by storing the index  $l_k$  and the vector  $y_k$ .

After  $k$  such updates we have

$$B_k = B_0T_1T_2 \cdots T_k. \quad (3.12)$$

Given a procedure to solve systems of equations involving  $B_0$ , (3.12) indicates that solving  $B_kv = b$  is equivalent to solving the  $k+1$  linear systems

$$B_0v_0 = b, \quad T_1v_1 = v_0, \quad \dots, \quad T_kv_k = v_{k-1}, \quad (3.13)$$

where the systems involving  $T_j$  are easy to solve. As  $k$  increases, the solution process becomes progressively more protracted, and the storage required to store the updates is strictly increasing. Therefore it becomes worthwhile to compute a factorization of  $B_k$  from scratch. Most current systems use an initial triangular factorization  $B_0 = L_0U_0$  (see Section 3.4), and recompute the factorization after  $k$  updates (typically  $k \leq 50$ ).

The PF update has two important advantages for sparse problems. First, the vectors  $\{y_j\}$  may be stored in a single sequential file, so that implementation is straightforward. Second, any

advance in the methods for linear equations is immediately applicable to the factorisation of  $B_0$ , since the update does not alter the initial factorisation. Thus,  $B_0$  may be represented by a "black box" procedure for solving equations (involving both  $B_0$  and  $B_0^T$ ).

Unfortunately, the PF update has two significant deficiencies. It is numerically unreliable if  $|e_k^T y_k|$  is too small (since  $T_k$  is then ill-conditioned), and the growth of data defining the updates is significantly greater than for alternative schemes.

**3.5.2. The Bartels-Golub update.** The instability of the PF update was first made prominent by Bartels and Golub (1969), who showed as an alternative that an  $LU$  factorisation can be updated in a stable manner (see also Bartels, Golub and Saunders, 1970; Bartels, 1971). Given an initial factorisation  $B_0 = L_0 U_0$ , the updates to  $L$  are represented in product form, but the sparse triangular matrix  $U$  is stored (and updated) explicitly. Thus, instead of the form (3.12) we have

$$B_k = L_0 T_1 T_2 \cdots T_k U_k \equiv L_k U_k, \quad (3.14)$$

where each  $T_j$  represents an update whose construction will be discussed below.

At the  $k$ -th step, replacing the  $l_k$ -th column of  $B_{k-1}$  gives

$$B_k = L_{k-1} \tilde{U},$$

where  $\tilde{U}$  is identical to  $U_{k-1}$  except for its  $l_k$ -th column. Since  $U_{k-1}$  is stored as a sparse matrix, it is desirable to restore  $\tilde{U}$  to upper-triangular form  $U_k$  without causing substantial fill-in. To this end, let  $P$  denote a cyclic permutation that moves the  $l_k$ -th row and column of  $\tilde{U}$  to the end, and shifts the intervening rows and columns forward. We then have

$$\tilde{U} = \begin{array}{|c|} \hline \text{Sparse Matrix} \\ \hline \end{array}, \quad P^T \tilde{U} P = \begin{array}{|c|} \hline \text{Upper-triangular Matrix} \\ \hline \end{array}.$$

The nonzeros in the bottom row of  $P^T \tilde{U} P$  may be eliminated by adding multiples of the other rows. However, it follows from the usual error analysis of Gaussian elimination (e.g., Wilkinson, 1965) that this procedure will not be numerically stable unless the size of the multiple is bounded in some way. Hence, we must allow the last row to be interchanged with some other row. Formally, the row operations are stabilized elementary transformations (Wilkinson, 1965), which are constructed from  $2 \times 2$  matrices of the form

$$M = \begin{pmatrix} 1 & \\ \mu & 1 \end{pmatrix} \quad \text{or} \quad \tilde{M} = \begin{pmatrix} & 1 \\ 1 & \mu \end{pmatrix}. \quad (3.15)$$

(Note that the transformation  $\tilde{M}$  includes a row interchange.) Each such transformation is represented by the scalar  $\mu$ , and is unnecessary if the element to be eliminated is already zero. Numerical stability is achieved by choosing between  $M$  and  $\tilde{M}$  so that the multiplier  $\mu$  is bounded in size by some moderate number (e.g.,  $|\mu| \leq 1, 10$  or  $100$ ). The matrices  $\{T_j\}$  in (3.14) are constructed from sequences of matrices of the form (3.15).

Unfortunately, elimination of the nonzeros is "easier said than done" in the sparse case. Any transformation of type  $\tilde{M}$  amounts to a form of fill-in, since the location of nonzeros in the interchanged rows is unlikely to be the same. A complex data structure is therefore needed to update  $U_k$  without losing efficiency during subsequent solves. (Holding individual nonzeros in a linked list, for example, would not be acceptable in a virtual-memory environment.)

The implementation of the BG update by Saunders (1976) capitalizes on the "bump and spike" structure revealed by the  $P^4$  procedure (see Section 3.4). Each triangular factor is of the form

$$U_k = \begin{pmatrix} I & E_k \\ & F_k \end{pmatrix},$$

and fill-in can occur only within  $F_k$ . If  $U_0$  contains  $s$  spikes, the dimension of  $F_k$  will be at most  $s + k$ . Storing  $F_k$  as a dense matrix allows the BG update to be implemented with maximum stability ( $|\mu| \leq 1$  in (3.15)), and the approach is efficient as long as  $s$  is not unduly large (say,  $s \leq 100$ ). This implementation has been used for several years in the nonlinear programming system MINOS (Murtagh and Saunders, 1977, 1980). During that period, the number of spikes in  $U_0$  has proved to be favorably small for many sparse optimization models. However, two important applications are now known to give unacceptably large numbers of spikes: time-period models (for which  $B$  has a staircase structure) and optimal power-flow problems (for which  $B$  has a symmetric sparsity pattern). Some statistics for these problems are given in Table 1 (Section 3.4).

Another implementation of the BG update has been developed by Reid (1976, 1982) as the Fortran package LA05 in the Harwell Subroutine Library. It strikes a compromise between dense and linked-list storage by using a whole row or column of  $U_k$  as the "unit" of storage. Thus, the nonzeros in any one row of  $U_k$  are held in contiguous locations of memory, as are the corresponding column indices, and an ordered list points to the beginning of each row. To facilitate searching, a similar data structure is used to hold just the sparsity pattern of each column (i.e., the row indices are stored, but not the nonzeros themselves). This storage scheme is also suitable for computing an initial  $LU$  factorization using the Markowitz criterion and threshold pivoting — a combination that has been eminently successful in practice, particularly on the structures mentioned above. Table 2 (Section 3.4) shows the small increase in the number of nonzeros using LA05.

We note that in the context of column updating, the stability test in the initial factorization should ideally be performed along the columns of  $L_0$ , rather than along the rows of  $U_0$  as in the existing LA05, in order to ensure that  $L_0$  in (3.14) is well conditioned. (This is necessary to achieve the following desirable property: the factorization of  $B_k$  is likely to be well conditioned if  $B_k$  is well conditioned, even if  $B_0$  is not.) For efficiency the data structure for computing  $U_0$  then needs to be transposed. This and other improvements will be incorporated in a new version of LA05 (Reid, private communication).

In the meantime, the sparsity properties of LA05 are unsurpassed, and the numerical properties are excellent as long as  $B_0$  is not extremely ill-conditioned. The package should therefore find increasingly widespread application.

**3.5.3. The Forrest-Tomlin update.** The update of Forrest and Tomlin (1972) was developed as a means of improving upon the sparsity of the PF update while retaining the ability to use external storage where necessary. In fact the FT update is a restricted form of the BG update, in which no row interchanges are allowed when eliminating the bottom row of  $P^T \tilde{U} P$ . This single difference removes the fill-in difficulty (but at the expense of losing guaranteed numerical stability).

Algebraically, a new column  $w_k$  is added to  $U_{k-1}$ , the  $l_k$ -th column and row are deleted, and the transformations  $M$  are combined into a single "row" transformation  $R_k = I + e_{l_k}(r_k - e_{l_k})^T$ .

It can be shown that the required vectors satisfy

$$L_{k-1}w_k = v_k, \quad \text{and} \quad U_{k-1}^T r_k = e_{l_k}, \quad (3.16)$$

and the new diagonal of  $U_k$  is  $r_k^T w_k$ . Most importantly, the multipliers  $\mu$  are closely related to the elements of  $r_k$ , and these can be tested *a posteriori* to determine whether the update is acceptable (see also Tomlin, 1975). In practice a rather undemanding test such as  $|\mu| \leq 10^6$  must be used to avoid rejecting the update too frequently. The FT update is now used within several commercial mathematical programming systems.

**3.5.4. Use of the Schur complement.** The work of Bisschop and Meeraus (1977, 1980) has recently provided a new perspective on the problem of updating within active-set methods. Suppose that for each update a vector  $v_j$  replaces the  $l_j$ -th column of  $B_0$ . A key observation is that the system  $B_k x = b$  is equivalent to the system

$$\begin{pmatrix} B_0 & V_k \\ I_k & \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad (3.17)$$

where

$$V_k = (v_1 \ v_2 \ \cdots \ v_k), \quad I_k = (e_{l_1} \ e_{l_2} \ \cdots \ e_{l_k})^T.$$

Note that the rectangular matrix  $I_k$  is composed of  $k$  rows of the identity matrix corresponding to indices of columns that have been replaced. Since the equations  $I_k y = 0$  set  $k$  elements of  $y$  to zero, the remaining elements of  $y$  and  $z$  together give the required solution  $x$ . Similarly, the system  $B_k^T y = d$  is equivalent to

$$\begin{pmatrix} B_0^T & I_k^T \\ V_k^T & \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad (3.18)$$

if  $d_1$  and  $d_2$  are constructed from  $d$  appropriately (with the aid of  $k$  arbitrary elements, such as zero).

The matrix in (3.17) may be factorized in several different ways. In the next two sections we consider the simplest factorization

$$\begin{pmatrix} B_0 & V_k \\ I_k & \end{pmatrix} = \begin{pmatrix} B_0 & \\ I_k & C_k \end{pmatrix} \begin{pmatrix} I & Y_k \\ & I \end{pmatrix} \quad (3.19)$$

where

$$\begin{aligned} B_0 Y_k &= V_k, \\ C_k &= -I_k Y_k. \end{aligned} \quad (3.20)$$

The  $k \times k$  matrix  $C_k$  is the Schur complement for the partitioned matrix on the left-hand side of (3.19). It corresponds to a matrix of the ubiquitous form  $D - WB^{-1}V$  (e.g., see Cottle, 1974).

**3.5.5. A stabilised product-form update.** From (3.17) and (3.19) we see that the vectors  $y$  and  $z$  needed to construct the solution of  $B_k x = b$  may be obtained from the equations

$$B_0 w = b, \quad (3.21a)$$

$$C_k x = -w, \quad (3.21b)$$

$$y = w - Y_k x. \quad (3.21c)$$

Similarly, the solution of  $B_k^T y = d$  is obtained from the two linear systems

$$C_k^T x = d_2 - Y_k^T d_1, \quad (3.22a)$$

$$B_0^T y = d_1 - I_k^T x. \quad (3.22b)$$

Assuming that  $Y_k$  is available, the essential operations in (3.21) and (3.22) are a solve with  $B_0$  and a solve with  $C_k$ . If  $k$  is small enough (say,  $k \leq 100$ ),  $C_k$  may be treated as a dense matrix. It is then straightforward to use an orthogonal factorization  $Q_k C_k = R_k$  ( $Q_k^T Q_k = I$ ,  $R_k$  upper triangular) or an analogous factorization  $L_k C_k = U_k$  based on Gaussian elimination ( $L_k$  square,  $U_k$  upper triangular). These factorizations can be maintained in a stable manner as  $C_k$  is updated to reflect changes to  $B_k$ . (The updates involve adding and deleting rows and columns of  $C_k$ ; see Gill et al., 1974.) The stability of the procedures (3.21) and (3.22) then depends essentially on the condition of  $B_0$ . In other words, if  $B_0$  is well conditioned, we have a stable method for solving  $B_k x = b$  for many subsequent  $k$ .

This method retains several advantages of the PF update. The vectors to be stored (columns of  $Y_k$ ) satisfy  $B_0 y_k = v_k$ , which is analogous to (3.11). These vectors should have sparsity similar to those in the PF update, and they can be stored sequentially (in compact form on an external file, if necessary). A further advantage is that whenever a column of  $C_k$  is deleted, the corresponding vector  $y_k$  may be skipped in subsequent uses of (3.21c). This gain would tend to offset the work involved in maintaining the factors of  $C_k$ . Because of the parallels, the method described here amounts to a practical mechanism for stabilizing an implementation based on the PF update.

**3.5.6. The Schur-complement update.** One of the aims of Bisschop and Meeraus (1977, 1980) was to give an update procedure whose storage requirements were independent of the dimension of  $B_0$ . This is achievable because the matrix  $Y_k$  is not essential for solving (3.17) and (3.18), given  $V_k$  and a "black box" for  $B_0$ . For example, (3.21c) may be replaced by

$$B_0 y = b - V_k z, \quad (3.23),$$

and hence storage for  $Y_k$  can be saved at the expense of an additional solve with  $B_0$ . Similarly, (3.22a) is equivalent to

$$\begin{aligned} B_0^T w &= d_1, \\ C_k^T x &= d_2 - V_k^T w, \end{aligned}$$

again involving a second solve with  $B_0$ . Note that the original data  $V_k$  will usually be more sparse than  $Y_k$ , so that the additional expense may not be substantial.

The storage required for a dense orthogonal factorization of  $C_k$  ( $\frac{1}{2}k^2$ ) is small for moderate values of  $k$ . As with the PF update, any advance in solving linear equations is immediately applicable to the equations involving  $B_0$ .

The method is particularly attractive when  $B_0$  has special structure. For example, certain linear programs have the following form:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && (B_0 \quad N)x = b \\ &&& l \leq x \leq u, \end{aligned}$$

where  $B_0$  is a square block-diagonal matrix:

$$B_0 = \text{block-diag}(D_0 \ D_1 \ \cdots \ D_N).$$

Assuming that the square matrices  $D_j$  are well conditioned,  $B_0$  provides a natural starting basis for the simplex method.

With the Schur-complement (SC) update, an iteration of the simplex method on such a problem requires four solves with  $B_0$ , and hence four solves with each matrix  $D_j$ . In certain applications, the matrices  $D_j$  are closely related to  $D_0$  (e.g., in time-dependent problems), in which case a further application of the Schur-complement technique would be appropriate. A simplex iteration then involves only solves with  $D_0$ .

This is a situation in which one factorization is followed by hundreds or even thousands of solves (involving both  $D_0$  and  $D_0^T$ ). Thus, it is useful for black-box solvers to be tuned to the case of multiple right-hand sides.

**3.5.7. The partitioned LU update.** Recall that the PF approach accumulates updates in a single file, while the BG and FT methods seek to reduce the storage required for the updates by updating two separate factors (one implicitly through a file of updates, the other explicitly). Here we suggest leaving  $L_0$  and  $U_0$  unaltered (in effect, treating them as two "black boxes" for solving linear systems), and accumulating two files of updates. In place of the block factorization (3.19) we can write

$$\begin{pmatrix} B_0 & V_k \\ I_k & \end{pmatrix} = \begin{pmatrix} L_0 & \\ R_k & C_k \end{pmatrix} \begin{pmatrix} U_0 & W_k \\ & I \end{pmatrix} \quad (3.24)$$

with the same definition (3.20) of  $C_k$ . After the  $k$ -th update, the new column of  $W_k$  and row of  $R_k$  satisfy

$$L_0 w_k = v_k \quad \text{and} \quad U_0^T r_k = e_{i_k}. \quad (3.25)$$

The similarity of (3.25) with the equations (3.16) for the FT update leads us to suppose that the storage requirements would be at least as low as for the FT update. Apart from the need to store and update  $C_k$ , all implementation advantages are retained (in fact improved upon, since  $U_0$  is not altered). As with the PF and SC updates, the stability depends primarily on the condition of  $B_0$ . We could therefore regard the factorization (3.24) as a practical and stable alternative to the FT update.

**3.5.8. Avoiding access to  $B_0$ .** In active-set methods, it is often necessary to solve the equations  $B_k x = v$ , where  $v$  is a column of the matrix  $A$ . Although  $v$  will not be a column of  $B_k$ , it could be a column of  $B_0$ . If  $B_0$  were not stored in main memory, it would be desirable to access its columns as seldom as possible. In this section we shall show that with the PF update or the Schur-complement updates, the elements of  $B_0$  need not be accessed once the initial factorization has been completed.

Assume that  $v$  is the  $l$ -th column of  $B_0$ , so that  $v = B_0 e_l$  by definition. For the PF update it follows by substituting the expression for  $v$  in (3.13) that

$$T_1 \cdots T_k x = e_l,$$

which gives an equation for  $x$  that does not involve  $v$  or  $B_0$ . With the Schur-complement approach, (3.21a) reduces to  $w = e_l$ , while (3.23) can be rearranged to give  $B_0(y - e_l) = -V_k x$ . In either case, when solving for  $x$  we can avoid not only an explicit reference to the elements of  $B_0$  but also a solve with  $B_0$ .

Similarly, it is often necessary to solve  $B_k^T y = d$  and then to form  $\gamma_j = y^T v_j$  for each column  $v_j$  that has been replaced in  $B_0$ . (The quantities  $\gamma_j$  are the reduced costs or reduced gradients for variables that have been removed from  $B_0$ .) If  $t$  denotes the product  $B_0^T y$ , then by definition of  $v_j$  it follows that  $y^T v_j = t^T e_{i_j}$ . With both the PF and the Schur-complement updates,  $t$  is a by-product of the procedure for computing  $y$ . Thus,  $t$  and all relevant values  $\gamma_j$  are available at no cost.

These results confirm that  $B_0$  need exist only in the form of a "black box" for solving linear systems.

**3.6. Other applications of the Schur-complement update.** Historically, the formulation LCP has been used because it involves only column updates to  $B_k$ , which have appeared to be the least difficult kind of update to implement for sparse problems. However, the Schur-complement approach also applies to more general sequences of related square systems. As with column replacement, the key idea is to solve a partitioned system that involves the original matrix.

**3.6.1. Unsymmetric rank-one updates.** Consider the case in which  $B_0$  undergoes a sequence of rank-one modifications:

$$B_k = B_{k-1} + v_k s_k^T \equiv B_0 + V_k S_k^T.$$

The solution of  $B_k x = b$  is part of the solution of the extended system

$$\begin{pmatrix} B_0 & V_k \\ S_k^T & -I \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \quad (3.26)$$

(Kron, 1956; Bisschop and Meeraus, 1977). Given factorisations of  $B_0$  and the Schur complement  $C_k \equiv -I - S_k^T B_0^{-1} V_k$ , the solution may be obtained from

$$\begin{aligned} C_k z &= -S_k^T w, \\ B_0 x &= b - V_k z, \end{aligned}$$

where  $B_0 w = b$ . An alternative that would require more storage but less work could be obtained by using  $B_0 = L_0 U_0$  and storing the vectors defined by  $L_0 w_k = v_k$ ,  $U_0^T r_k = s_k$ . Let  $R_k$  denote the matrix whose  $j$ -th column is  $r_j$ , and similarly for  $W_k$ . In this case, the solution of (3.26) would be obtained from

$$\begin{aligned} C_k z &= -R_k^T v \\ U_0 x &= v - W_k z, \end{aligned}$$

where  $L_0 v = b$ . Either approach is an alternative to updating a factorization of  $B_k$  itself (e.g., Gille and Louie, 1981, 1982), which is even more difficult to implement than the BG update.

We emphasize that column or row replacements are best treated as a special case, not as a sequence of general rank-one modifications.

**3.6.2. A symmetric Schur-complement update.** It was observed in Section 3.1 that in some circumstances the search direction can be computed by solving the linear system (3.2) involving the augmented matrix

$$M_k = \begin{pmatrix} H_k & A_k^T \\ A_k & \end{pmatrix}. \quad (3.27)$$

Within an active-set method, changes in the status of fixed and free variables lead to changes in  $H$  and  $A$ . When a variable becomes fixed, the corresponding row and column of  $M_k$  are deleted; when a variable is freed, a new row and column of  $M_k$  are added.

Instead of updating a factorization of  $M_k$ , we can start with some  $M_0$  and work with an augmented system of the form

$$\begin{pmatrix} M_0 & S_k \\ S_k^T & \end{pmatrix}.$$

If a variable is fixed at the  $k$ -th change, the  $k$ -th column of  $S_k$  is an appropriate coordinate vector; if the  $l$ -th variable is freed, the column is

$$s_k = \begin{pmatrix} h_l \\ a_l \end{pmatrix},$$

where  $h_l$  is obtained from the  $l$ -th column of the full Hessian, and  $a_l$  is the  $l$ -th column of  $A$ . The solution of the augmented system corresponding to the  $k$ -th working set can then be obtained using a factorization of  $M_0$  and a factorization of the Schur complement  $C_k = -S_k^T M_0^{-1} S_k$ .

**3.7. Linear and quadratic programming.** Two important special cases of LCP are linear and quadratic programs. Since there are no user-supplied functions, the computation in linear and quadratic programming methods involves primarily linear algebraic operations.

**3.7.1. Large-scale linear programming.** Large-scale linear programs occur in many important applications, such as economic planning and resource allocation. Methods and software for large-scale LP have thus achieved a high level of sophistication, and many of the techniques discussed in Section 3 were designed originally for use within the simplex method.

Much research has involved linear programs with special structure in the constraint matrix — for example, those arising from networks or time-dependent systems. It is impossible to summarise methods for specially-structured linear programs in a survey paper of this type. However, to illustrate the flavor of the work, we consider staircase linear programs (which were used in the examples of Section 3.4). These arise in modeling time-dependent processes; the recent book edited by Dantsig, Dempster and Kallio (1981) is entirely devoted to such problems. It has long been observed that the simplex method tends to be less efficient on staircase problems than on general LPs. To correct this deficiency, work has tended to proceed in two directions. First, the simplex method can be adapted to take advantage of the staircase structure, by using special techniques for factorizing, updating, and pricing (Fourer, 1982). Second, special-purpose methods can be designed to exploit particular features of the problem. For staircase problems, several variations of the *decomposition* approach (Dantsig and Wolfe, 1960) have been suggested. The basic idea is to solve the problem in terms of smaller, nearly independent, subproblems.

**3.7.2. Large-scale quadratic programming.** A general statement of the quadratic programming problem is

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && c^T x + \frac{1}{2} x^T H x \\ & \text{subject to} && Ax = b \\ & && l \leq x \leq u, \end{aligned}$$

where  $H$  is a symmetric matrix.

An early approach to quadratic programming was to transform the problem into a linear program, which is then solved by a modified LP method (e.g., Beale, 1967). The most popular quadratic programming algorithms are now based on the active-set approach described in Section 3.1 (for a comprehensive survey of QP methods, see Cottle and Djang, 1979), and the search direction is defined by the subproblem (3.1). Efficient methods for *sparse* quadratic programs thus involve specializing the techniques discussed in Section 3.3 for the special case when the Hessian is constant.

#### 4. Nonlinearly Constrained Optimization

The nonlinearly constrained optimization problem is assumed to be of the following form:

$\begin{array}{ll} \text{NCP} & \text{minimise } F(x) \\ & \text{subject to } c(x) = 0 \\ & \ell \leq x \leq u \end{array}$
-----------------------------------------------------------------------------------------------------------------------------

where  $c(x)$  is a vector of  $m$  nonlinear constraint functions. We shall assume that these constraints are "sparse", in the sense that the  $m \times n$  Jacobian matrix  $A(x)$  of  $c(x)$  is sparse. For simplicity, we shall usually not distinguish between linear and nonlinear constraints in  $c(x)$ . However, it is usually considered desirable to treat linear and nonlinear constraints separately.

Problems with nonlinear constraints are considerably more difficult to solve than those with only linear constraints. There is an enormous literature concerning methods for nonlinear constraints; recent overviews are given in Fletcher (1981) and Gill, Murray and Wright (1981). In this section, we shall concentrate on the impact of sparsity rather than attempt a thorough discussion of the methods.

One aspect of NCP that is directly relevant to sparse matrix techniques is that any super-linearly convergent algorithm must consider the curvature of the nonlinear constraint functions, and thus the Hessian of interest is the Hessian of the *Lagrangian function* rather than the Hessian of  $F$  alone. Let the Hessian of the Lagrangian function be denoted by  $W(x, \lambda) \equiv H(x) - \sum_{i=1}^m \lambda_i H_i(x)$ , where  $H_i$  is the Hessian of  $c_i$ . At first, it might appear unlikely that this matrix would be sparse, since it is a weighted sum of the Hessians of the objective function and the constraints. However, sparsity in the gradient of a nonlinear constraint always implies sparsity in its Hessian matrix. For example, if the gradient of  $c_i(x)$  contains five nonzero components, the corresponding Hessian matrix  $H_i(x)$  can have at most 25 nonzero elements. Furthermore, there is often considerable overlap in the positions of nonzero elements in the Hessians of different constraints. Thus, *in practice the Hessian of the Lagrangian function is often very sparse*.

The usual approach to solving NCP is to construct a sequence of unconstrained or linearly constrained subproblems whose solutions converge to that of NCP. Early methods included unconstrained subproblems based on penalty and barrier functions (see Fiacco and McCormick, 1968). Unfortunately, these methods suffer from inevitable ill-conditioning; they have for the most part been superseded by more efficient methods.

**4.1. Augmented Lagrangian methods.** Augmented Lagrangian methods were motivated in large part by the availability of good methods for unconstrained optimization. The original idea was to minimise an approximation to the Lagrangian function that has been suitably augmented (by a penalty term) so that the solution is a local unconstrained minimum of the augmented function (Hestenes, 1969; Powell, 1969).

In particular, an augmented Lagrangian method can be defined in which  $x_{k+1}$  is taken as the solution of the subproblem

$$\begin{array}{ll} \text{minimise} & L_A(x, \lambda_k, \rho_k) \\ \text{subject to} & \ell \leq x \leq u, \end{array} \quad (4.1)$$

where the augmented Lagrangian function  $L_A$  is defined by

$$L_A(x, \lambda, \rho) \equiv F(x) - \lambda^T c(x) + \frac{\rho}{2} c(x)^T c(x). \quad (4.2)$$

The vector  $\lambda_k$  is an estimate of the Lagrange multiplier vector, and  $\rho_k$  is a suitably chosen non-negative scalar. Alternatively, it is possible to treat any general linear constraints by an active-set method (Section 3.1), and to include only *nonlinear* constraints in the augmented Lagrangian function. Whatever the definition of the subproblem, the algorithm has a *two-level* structure — “outer” iterations (corresponding to different subproblems) and “inner” iterations (within each subproblem).

The Hessian of interest when solving (4.1) is the Hessian of  $L_A$  (4.2), which is  $W(x, \lambda_k) + \rho_k A(x)^T A(x)$ . If the only constant elements of the Jacobian matrix are zero, the sparsity patterns of  $W(x, \lambda)$  and the Hessian matrix of  $L_A$  are generally identical. Hence, techniques designed to use an explicit sparse Hessian may be applied to (4.1).

The Jacobian matrix  $A(x)$  need not be stored explicitly in order to solve the subproblem (4.1). If a fairly accurate solution of (4.1) is computed, an improved Lagrange multiplier estimate may be obtained without solving any linear systems involving  $A(x)$ . However, in several recent augmented Lagrangian methods, (4.1) is solved only to *low accuracy* in order to avoid expending function evaluations when  $\lambda_k$  is a poor estimate of the optimal multipliers; in this case, some factorization of the matrix  $A(x_{k+1})$  is required to obtain an improved Lagrange multiplier estimate (by solving either a linear system or a linear least-squares problem). The relevance of the storage needed for the Jacobian and/or a factorization depends on the number of nonlinear constraints and the sparsity of the Jacobian.

**4.2. Linearly constrained subproblems.** The solution of NCP is a minimum of the Lagrangian function in the subspace defined by the gradients of the active constraints. This property leads to a class of methods in which linearizations of the nonlinear constraints are used to define a *linearly constrained subproblem*, of the form

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && F(x) - \lambda_k^T (c(x) - A_k^T x) \\ & \text{subject to} && A_k(x - x_k) = -c_k \\ & && \ell \leq x \leq u, \end{aligned} \tag{4.3}$$

where  $c_k$  and  $A_k$  denote  $c(x_k)$  and  $A(x_k)$  (Robinson, 1972; Rosen and Kreuser, 1972). With this formulation, the Lagrange multipliers of the  $k$ -th subproblem may be taken as the multiplier estimate  $\lambda_{k+1}$  in defining the next subproblem, and will converge to the true multipliers at the solution. When  $c(x)$  contains both linear and nonlinear functions, only the nonlinear functions need be included in the objective function of (4.3). Under suitable assumptions, the solutions of the subproblems converge *quadratically* to the solution of NCP. A further benefit of the subproblem (4.3) is that linear constraints may be treated explicitly.

One of the important conditions for convergence with the subproblems (4.3) is a “sufficiently close” starting point; thus, some procedure must be used to prevent divergence from a poor value of  $x_0$ . Rosen (1980) suggested a two-phase approach, starting with a penalty function method. In the MINOS/AUGMENTED system of Murtagh and Saunders (1982), the objective function of the subproblem is defined as a *modified augmented Lagrangian* of the form

$$\tilde{L}_A(x, \lambda_k, \rho_k) \equiv F(x) - \lambda_k^T \tilde{c}_k(x) + \frac{\rho_k}{2} \tilde{c}_k(x)^T \tilde{c}_k(x), \tag{4.4}$$

where

$$\tilde{c}_k(x) = c(x) - (c_k + A_k(x - x_k)).$$

The sparsity pattern of  $\tilde{L}_A$  is identical to that of  $W(x, \lambda)$ , irrespective of any nonzero constant elements in the Jacobian matrix.

Methods based on solving (4.3) have several benefits for sparse problems. The ability to treat linear constraints explicitly is helpful for the many large problems in which most of the constraints are linear. As noted in the Introduction, it is often a feature of sparse problems that the cost of evaluating the problem functions is dominated by the sparse matrix operations. The superiority of SQP methods (Section 4.3.2) for dense problems results from the generally lower number of function evaluations compared to methods based on (4.3); for sparse problems, however, the function evaluations required to solve (4.3) may be insignificant compared to the savings that would result from solving fewer subproblems. If an active-set method of the type described in Section 3.3.1 is applied to (4.3), only the *projected Hessian* needs to be stored (rather than the full Hessian). Thus, methods based on (4.3) will tend to be more effective than augmented Lagrangian methods for problems in which the Hessian of the Lagrangian function is not sparse and the projected Hessian can be stored as a dense matrix.

**4.3. Methods based on linear and quadratic programming.** We now consider two classes of methods in which the subproblems are solved *without evaluation of the problem functions* (in contrast to the methods of Sections 4.1 and 4.2).

**4.3.1. Sequential linear programming methods.** Because of the availability and high quality of software for sparse linear programs, a popular technique for solving large-scale problems has been to choose each iterate as the solution of an LP subproblem; we shall call these *sequential linear programming* (SLP) methods. They were first proposed by Griffith and Stewart (1961); for a recent survey, see Palacios-Gomez, Lasdon and Engquist (1982).

One crucial issue in an SLP method is the definition of the linear functions in the subproblem. A typical formulation is

$$\begin{aligned} &\underset{z \in \mathbb{R}^n}{\text{minimise}} && g_k^T(x - x_k) \\ &\text{subject to} && A_k(x - x_k) = -c_k \\ &&& \ell \leq x \leq u. \end{aligned}$$

With some formulations, the LP may not be well posed — for example, there may be fewer constraints than variables. The usual way of ensuring a correctly posed subproblem is to include additional constraints on the variables, such as bounds on the change in each variable. In general, the latter are also needed to ensure convergence.

SLP methods have the advantage that the subproblems can be solved using all the technology of sparse LP codes. They tend to be efficient on two types of problems: those with nearly linear functions, particularly slightly perturbed linear programs; and those in which the functions can be closely approximated by piecewise linear functions (e.g., the objective function is separable and convex). Unfortunately, on general problems SLP methods are at best linearly convergent unless the number of active constraints at the solution is equal to the number of variables. Furthermore, the speed of convergence critically depends on the technique that defines each subproblem.

Recently, some of the techniques used in SQP methods (Section 4.3.2) have been applied to the SLP approach — such as the use of a merit function to ensure progress after each outer iteration. Such techniques cannot be expected to improve the asymptotic rate of convergence of SLP methods, but they should improve robustness and overall effectiveness.

Beale (1978) has given a method that is designed to make extensive use of an existing LP system. The nonlinearly constrained problem is assumed to be of the form

$$\begin{aligned} & \underset{x,y}{\text{minimize}} && c(x)^T y \\ & \text{subject to} && A(x)y = b(x) \\ & && \ell \leq x \leq u \\ & && v \leq y \leq w. \end{aligned} \tag{4.5}$$

A special nonlinear algorithm is then used to adjust  $x$ ; for each value of  $x$ , a new estimate  $y$  is determined by solving an LP.

**4.3.2. Sequential quadratic programming methods.** The most popular methods in recent years for dense nonlinearly constrained problems are based on solving a sequence of quadratic programming subproblems (see Powell, 1982, for a survey). At iteration  $k$ , a typical QP subproblem has the form

$$\begin{aligned} & \underset{p \in \mathbb{R}^n}{\text{minimize}} && \frac{1}{2} p^T H_k p + g_k^T p \\ & \text{subject to} && A_k p = -c_k \\ & && \ell - x_k \leq p \leq u - x_k, \end{aligned}$$

where  $H_k$  is an approximation to the Hessian of the Lagrangian function. The solution of the QP subproblem is then used as the *search direction*  $p_k$  in (1.1). The step  $\alpha_k$  is chosen to achieve a suitable reduction in some *merit function* that measures progress toward the solution. In the dense case, the most popular method is based on taking  $H_k$  as a positive-definite quasi-Newton approximation to the Hessian (Powell, 1977). However, the many options in defining the QP subproblem have yet to be fully understood and resolved (see Murray and Wright, 1982, for a discussion of some of the critical issues).

Further complex issues are raised when applying an SQP method to sparse problems (see, e.g., Gill *et al.*, 1981). The general development of methods has been hampered because methods for sparse quadratic programming are only just being developed, and are not yet generally available for use within a general nonlinear algorithm. However, Escudero (1980) has reported some success with an SQP implementation in which a sparse quasi-Newton approximation is used for  $H_k$  (see also Section 3.7.2).

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**ABSTRACT - SOL 82-17****Sparse Matrix Methods in Optimization****by P.E. Gill, W. Murray, M.A. Saunders and M.H. Wright**

Optimization algorithms typically require the solution of many systems of linear equations  $B_k y_k = b_k$ . When large numbers of variables or constraints are present, these linear systems could account for much of the total computation time.

Both direct and iterative equation solvers are needed in practice. Unfortunately, most off-the-shelf solvers are designed for single systems, whereas optimization problems give rise to hundreds or thousands of systems. To avoid refactorization, or to speed the convergence of an iterative method, it is essential to note that  $B_k$  is related to  $B_{k-1}$ .

We review various sparse matrices that arise in optimization, and discuss compromises that are currently being made in dealing with them. Since significant advances continue to be made with single-system solvers, we give special attention to methods that allow such solvers to be used repeatedly on a sequence of modified systems (e.g., the product-form update; use of the Schur complement). The speed of factorizing a matrix then becomes relatively less important than the efficiency of subsequent solves with very many right-hand sides.

At the same time, we hope that future improvements to linear-equation software will be oriented more specifically to the case of related matrices  $B_k$ .